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A Method for Converting a Class of Univariate Functions into d.c. Functions

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Abstract. D.c. functions are functions that can be expressed as the sum of a concave function and a convex function (or as the difference of two convex functions). In this paper, we extend the class of univariate functions that can be represented as d.c. functions. This expanded class is very broad including a large number of nonlinear and/or 'nonsmooth' univariate functions. In addition, the procedure specifies explicitly the functional and numerical forms of the concave and convex functions that comprise the d.c. representation of the univariate functions. The procedure is illustrated using two numerical examples. Extensions of the conversion procedure for discontinuous univariate functions is also discussed.

Key words: D.c. functions, D.c. optimization, Conversion procedure

1. Introduction

In this paper, we consider optimization problems of the form

$$\text{Problem } \mathscr{P}: \qquad \text{global min}\,\{f(\underline{\boldsymbol{x}}) \quad \text{s.t.} \quad \underline{\boldsymbol{x}} \in X\} \tag{1}$$

where $\underline{\boldsymbol{x}} = (x_1, \ldots, x_n)^\mathsf{T} \in \mathbb{R}^n$ is a n-dimensional decision variable vector, $X \subset \mathbb{R}^n$ is a compact, convex set, $f(\underline{\boldsymbol{x}}) = \sum_{i=1}^n f_i(x_i)$ is a separable real-valued function, and each univariate function $f_i : D_i \to \mathbb{R}$ is defined on a closed interval $D_i = [a_i, b_i]$ with endpoints a_i and b_i. The difficulty in solving problem $\mathscr{P}$ depends, in part, on the form of the functions $f_i(x_i)$. We consider four, progressively harder, cases.

First, if each $f_i(x_i)$ is convex for $x_i \in D_i$, then problem $\mathscr{P}$ is a convex minimization problem. In this case, a local minimum is also a global minimum and very efficient methods exist for solving problem $\mathscr{P}$. See, for example [1, 3, 5, 7, 8, 13, 15, 16, 19, 21] for solution methods for convex minimization problems.

Second, if each $f_i(x_i)$ is concave for $x_i \in D_i$, then problem $\mathscr{P}$ is a concave minimization problem and a global minimum will be at an extreme point of the feasible region of problem $\mathscr{P}$. For concave minimization problems, the global minimum can be found by methods that involve the relaxation of the objective function $f_i(x_i)$ and the partitioning of the feasible region X. Optimization methods for concave minimization problems are described in [2, 10–12, 17, 20].

Third, if each $f_i(x_i)$ can be expressed as*

$$f_i(x_i) = p_i(x_i) + q_i(x_i) \tag{2}$$

where, for each i, $p_i : D_i \to \mathbb{R}$ is a univariate *concave* function on D_i and $q_i : D_i \to \mathbb{R}$ is a univariate *convex* function on D_i, then $f_i(x_i)$ is called a 'd.c. function on D_i' and problem $\mathcal{P}$ is a d.c. minimization problem. This type of d.c. minimization problem can be converted to a concave minimization problem by reexpressing problem $\mathcal{P}$ as

$$\text{global min} \left\{ \sum_{i=1}^{n} p_i(x_i) + z \quad \text{s.t.} \quad \underline{\boldsymbol{x}} \in X \text{ and } \sum_{i=1}^{n} q_i(x_i) \leqslant z \right\} \tag{3}$$

where $z \in \mathbb{R}$ (see, for example, [2]). Recent surveys on the theory and application of d.c. optimization problems are contained in [11, 12, 22, 23].

Fourth, the hardest case is where each $f_i(x_i)$ is an arbitrary function that is, in general, neither concave nor convex for $x_i \in D_i$. Examples of this type of problem include distribution problems involving the joint production and transportation of goods, economic planning problems involving both decreasing and increasing marginal costs, and inventory control problems involving congestion effects as well as economies of scale.

The focus of this paper is on the last, and hardest, case described above. The paper makes two contributions. First, it enlarges the class of functions $f_i(x_i)$ that can be converted to a d.c. function (see (2)). Second, it provides a straightforward method of explicitly specifying the functional form of the concave function $p_i(x_i)$ and the convex function $q_i(x_i)$ in the d.c. function representation of $f_i(x_i)$. Once each $p_i(x_i)$ and $q_i(x_i)$ has been specified using the techniques given in this paper, problem $\mathcal{P}$ can be solved via the minimization given in (3) using established techniques for concave minimization problems. Thus, the material given in this paper can be viewed as a 'preprocessing' step in the solution procedure for problem $\mathcal{P}$.

This paper is organized as follows. Section 2 describes the extended class of functions that can be expressed as d.c. functions. Section 3 presents a procedure for explicitly specifying the concave and convex functions in the d.c representation of a function. Section 4 illustrates the procedure using two numerical examples. Finally, Section 5 summarizes the paper and discusses extensions to the procedure.

2. Extended class of functions

This section extends the class of functions that can be represented as d.c. functions. In the procedure described in this paper, we consider each function $f_i(x_i)$ in problem $\mathcal{P}$ separately, first for $i = 1$, then for $i = 2$, and so on up to $i = n$. Thus, in this discussion, we assume that we are focusing of the function $f_i(x_i)$ for a *particular*

* Equivalently, a d.c. function can be defined as the difference betwen two convex functions.

index i. Therefore, for convenience, we omit the subscript i in the remainder of this paper. This means that the definition of a d.c. function given in (2) is expressed as

$$f(x) = p(x) + q(x) \tag{4}$$

where $f : D \to \mathbb{R}$ is a univariate function defined on the interval $D = [a, b]$, $p : D \to \mathbb{R}$ is a univariate *concave* function on D, and $q : D \to \mathbb{R}$ is a univariate *convex* function on D.

To highlight the contributions of this paper, we first briefly review some previous results for d.c. functions. We say that a function $f(x)$ is of 'class C^2 on D' if the second derivatives of $f(x)$ are continuous for all $x \in D$; and we say that $f(x)$ is of 'class P-L on D' if $f(x)$ is a piecewise-linear continuous function for all $x \in D$. If $f(x)$ is of class C^2 on D or class P-L on D, then there exists a concave function $p(x)$ and a convex function $q(x)$ that satisfies (4) (see [6, 9, 14]). Although this is a useful result, it does not in and of itself provide a specification of the functional or numerical form for $p(x)$ and $q(x)$. For instance, consider the function

$$f(x) = 2 - e^{-10(x-2)^2} - e^{-0.1(x-5)^2} \quad \text{for } x \in D = [0, 10] \tag{5}$$

(see Figure 1). This function is of class C^2. But, it is not immediately obvious how the concave function $p(x)$ and the convex function $q(x)$ should be expressed take in order to satisfy (4).

Another result applies to any continuous function. If $f(x)$ is a continuous function on D, then for any $\epsilon > 0$ there exists a function $g(x)$ of class C^2 on D or class P-L on D (and therefore a d.c. function on D) such that

$$|f(x) - g(x)| \leqslant \epsilon \tag{6}$$

for all $x \in D$ (see, for example, [4], p. 133). Thus, if $f(x)$ is continuous, there exists a concave function $p(x)$ and a convex function $q(x)$ such that $p(x) + q(x)$ is arbitrarily close to $f(x)$ for all $x \in D$. For example, the continuous function

$$f(x) = \left|x^3 - x\right| \quad \text{for } x \in D = [-2, 2] \tag{7}$$

(see Figure 2), which is of neither class C^2 nor class P-L, can, in principle, be approximated sufficiently closely by a high-order polynomial $g(x)$ using the Weierstrass Approximation Theorem from calculus. However, this approximation is of limited use since

- it is not an exact representation of $f(x)$,
- high-order polynomials are, in general, difficult to evaluate,
- neither the functional nor the numerical forms of $g(x)$ are given, and
- it is not clear what the form of $p(x)$ and $q(x)$ should be in order to represent $g(x)$ as a d.c. function.

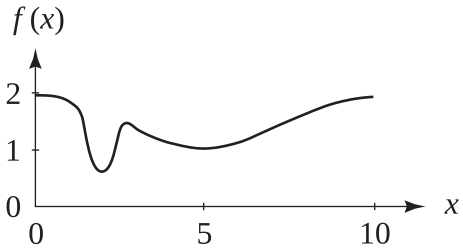


Figure 1. Function $f(x) = 2 - e^{-10(x-2)^2} - e^{-0.1(x-5)^2}$ for $D = [0\,,\,10]$.

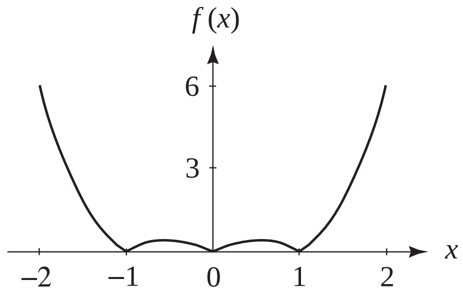


Figure 2. Function $f(x) = \left|x^3 - x\right|$ for $D = [-2\,,\,2]$.

We now present the contribution of this paper by describing a class of continuous functions (which is larger than class C^2 and class P-L) for which there is an exact d.c. representation. To define this class, for any function $f(x)$, let the first and second derivatives of $f(x)$ be denoted by $f'(x)$ and $f''(x)$, respectively. In addition, let the 'left-hand' (resp., 'right-hand') derivative of $f(x)$ be denoted by $f'_-(x)$ (resp., $f'_+(x)$).

DEFINITION 1. Let $f : D \to \mathbb{R}$ be a continuous function defined on a closed interval $D = [a\,,\,b]$. Then, the function $f(x)$ is said to be of 'class piecewise-C^2 on D' (or 'class P-C^2 on D', for short) if

- $f''(x)$ is continuous for all except a finite number of points in the open interval $(a\,,\,b)$; and
- $f'_-(x)$ and $f'_+(x)$ are finite for all points in the open interval $(a\,,\,b)$.

Note that class P-C^2 contains the functions of class C^2 as well as the functions of class P-L. However, class P-C^2 also contains functions which are neither class C^2 nor class P-L. For example, the nonlinear and 'nonsmooth' function $f(x)$ shown in Figure 2 is in class P-C^2.

In the next section, we show that any class P-C^2 function can be represented as a d.c. function. Thus, not only can the 'smooth' functions of the type shown in Figure 1 be represented exactly as a d.c. function, but so can the nonlinear and 'nonsmooth' functions of the type shown in Figure 2. Furthermore, for any class P-C^2 function $f(x)$, the form of the concave function $p(x)$ and the convex function $q(x)$ that satisfy (4) can be specified explicitly. Therefore, this method can be used to produce the functional and numerical forms of the concave and convex functions used in the d.c. representation of the functions shown in both Figures 1 and 2.

3. Conversion procedure

The purpose of this section is to describe a method for constructing a concave function $p(x)$ and a convex function $q(x)$ that satisfy (4) for any function $f(x)$ of class P-C^2 defined on a closed interval $D = [a\,, b]$. The conversion procedure – consisting of four steps – is given below.

The first step of the procedure is to partition the interval $D = [a\,, b]$ into a finite number of contiguous subintervals such that $f(x)$ alternates between a concave function and a convex function on adjacent subintervals. For instance, for the function shown in Figure 2, defined on the interval $D = [-2\,, 2]$, $f(x)$ is convex in the subinterval $[-2\,, -1]$, concave in the subinterval $[-1\,, 0]$, convex in the subinterval $[0\,, 0]$, concave in the subinterval $[0\,, 1]$, and so on.

To describe the partitioning procedure formally, let j be an index of the subintervals for $j = l, l+1, \ldots, r-1, r$ where l (resp., r) is the index of the leftmost (resp., rightmost) subinterval. Let $D_j = [a_j\,, b_j]$ denote the jth subinterval where a_j and b_j are endpoints of the jth subinterval.* The subintervals are contiguous and their union spans the original interval $D = [a\,, b]$. Thus, we have

$$a = a_l \leqslant b_l = a_{l+1} \leqslant b_{l+1} \quad \cdots \quad a_{r-1} \leqslant b_{r-1} = a_r \leqslant b_r = b \tag{8}$$

The subintervals are defined as follows:

DEFINITION 2. Let $f : D \to \mathbb{R}$ be a class P-C^2 function defined on a closed interval $D = [a\,, b]$ and let $D_j = [a_j\,, b_j]$ be the 'jth subinterval of D' for $j = l, l+1, \ldots, r-1, r$ where a_j and b_j satisfy (8). Then, the subintervals D_j are chosen such that the number of subintervals (i.e., $r - l + 1$) is as small as possible given that the following two conditions are satisfied:

- If j is odd, then for all $x \in (a_j\,, b_j)$, $f'_-(x)$ is nonincreasing, $f'_+(x)$ is nonincreasing, and $f'_-(x) \geqslant f'_+(x)$; and
- If j is even, then for all $x \in (a_j\,, b_j)$, $f'_-(x)$ is nondecreasing, $f'_+(x)$ is nondecreasing, and $f'_-(x) \leqslant f'_+(x)$.

In other words, if j is odd then $f(x)$ is concave on D_j; and if j is even then $f(x)$ is convex on D_j. By convention, if $f(x)$ is concave on the leftmost subinterval,

* Because we are supressing the subscript i, we denote the interval for the ith decision variable as D (rather than D_i). Similarly, we denote the jth subinterval for the ith decision variable as D_j (rather than D_{ij}).

then we set $l = 1$; whereas, if $f(x)$ is convex on the leftmost subinterval, then we set $l = 2$. Note that the definition of the subintervals admits the possibility that a_j may equal b_j for some of the subintervals. For instance, for the function shown in Figure 2, we have $D_2 = [-2\,, -1]$, $D_3 = [-1\,, 0]$, $D_4 = [0\,, 0]$, $D_5 = [0\,, 1]$, and $D_6 = [1\,, 2]$. The subinterval $D_4 = [0\,, 0]$ is included so that $f(x)$ is concave on D_j for j odd and convex on D_j for j even.

The method of determining the subintervals for a function is illustrated further in Appendix A.

The second step of the conversion procedure is to define a coefficient, denoted Δ_j, measuring the 'slope' of $f(x)$ at the right endpoint of the subinterval D_j . This information is obtained from the left-hand and right-hand derivatives which, as assumed for a class P-C^2 function, are finite-valued for all $x \in (a_j\,, b_j)$. For $j = l, l+1, \ldots, r-1$, the coefficient Δ_j is defined as

$$\Delta_j = \begin{cases} \min\left\{f'_-(b_j)\,,\, f'_+(b_j)\right\} & \text{if } j \text{ odd} \\ \max\left\{f'_-(b_j)\,,\, f'_+(b_j)\right\} & \text{if } j \text{ even} \end{cases} \tag{9}$$

The third step of the procedure is to define two affine functions $s_j : D \to \mathbb{R}$ and $t_j : D \to \mathbb{R}$. For $j = l, l+1, \ldots, r-1$, these functions are given by

$$s_j(x) = f(b_j) + \Delta_j \cdot (x - b_j) \tag{10}$$

$$t_j(x) = \sum_{k=l}^{j} (-1)^{j+k} \cdot s_k(x) \tag{11}$$

By definition, we set $s_j(x) \equiv 0$ and $t_j(x) \equiv 0$ if $j < l$.

The fourth and final step of the conversion procedure is to specify the functions $p : D \to \mathbb{R}$ and $q : D \to \mathbb{R}$. These functions are given by

$$p(x) = \begin{cases} f(x) - t_{j-1}(x) & \text{if } x \in D_j \text{ and } j \text{ odd} \\ t_{j-1}(x) & \text{if } x \in D_j \text{ and } j \text{ even} \end{cases} \tag{12}$$

$$q(x) = \begin{cases} t_{j-1}(x) & \text{if } x \in D_j \text{ and } j \text{ odd} \\ f(x) - t_{j-1}(x) & \text{if } x \in D_j \text{ and } j \text{ even} \end{cases} \tag{13}$$

The main results of the conversion procedure given above are summarized in the following three lemmas.

LEMMA 1. *Let $f : D \to \mathbb{R}$ be a class P-C^2 function defined on a closed interval $D = [a\,, b]$ and let the subintervals $D_j = [a_j\,, b_j]$ for $j = l, l+1, \ldots, r-1, r$ be specified according to Definition 2. Then, the number of subintervals (i.e., $r-l+1$) is finite.*

LEMMA 2. *Let $f : D \to \mathbb{R}$ be a class P-C^2 function defined on a closed interval $D = [a, b]$ and let $p(x)$ and $q(x)$ be specified according to* (12) *and* (13)*, respectively. Then, $p(x)$ is a concave function for all $x \in D$, and $q(x)$ is a convex function for all $x \in D$.*

LEMMA 3. *Let $f : D \to \mathbb{R}$ be a class P-C^2 function defined on a closed interval $D = [a, b]$ and let $p(x)$ and $q(x)$ be specified according to* (12) *and* (13)*, respectively. Then, $f(x) = p(x) + q(x)$ for all $x \in D$.*

The proof of these lemmas is contained in Appendix B. These lemmas state the desired result of this conversion procedure. Namely, if $f(x)$ is a class P-C^2 function, then it can be expressed as the sum of a concave function and a convex function.

The next section illustrates the mechanics of this conversion procedure.

4. Numerical examples

In this section, the functions shown in Figures 1 and 2 are used to illustrate the conversion procedure described in Section 3.

4.1. FIRST EXAMPLE

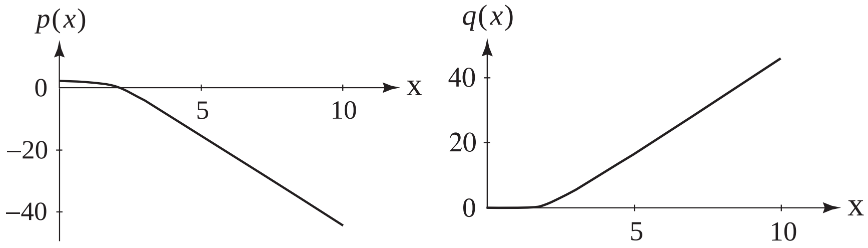
The function $f(x)$ shown in Figure 1 is given by (5). Since it is a class C^2 function, the first and second derivatives exit. They are given by

$$f'(x) = 20(x-2)e^{-10(x-2)^2} + 0.2(x-5)e^{-0.1(x-5)^2}$$

$$\begin{aligned} f''(x) &= 20\big(1 - 20(x-2)^2\big)e^{-10(x-2)^2} \\ &\quad + 0.2\big(1 - 0.2(x-5)^2\big)e^{-0.1(x-5)^2} \end{aligned}$$

The endpoints of the subintervals D_j are determined numerically by the endpoints of the original interval $D = [0, 10]$ and by the points where $f''(x)$ switches between a negative-valued function and a positive-valued function. Moreover, the slopes Δ_j are taken simply as the value of $f'(x)$ evaluated at the right endpoint of each subinterval since $f(x)$ is a class C^2 function and thus $f'_-(x) = f'_+(x) = f'(x)$. This yields

$$\begin{aligned} &D_1 = [a_1, b_1] = [0, 1.777095] && \Delta_1 = -2.940585 \\ &D_2 = [a_2, b_2] = [1.777095, 2.223145] && \Delta_2 = 2.455613 \\ &D_3 = [a_3, b_3] = [2.223145, 2.980195] && \Delta_3 = -0.267318 \\ &D_4 = [a_4, b_4] = [2.980195, 7.236068] && \Delta_4 = 0.271249 \\ &D_5 = [a_5, b_5] = [7.236068, 10] && \end{aligned}$$



(a) Concave function $p(x)$ (b) Convex function $q(x)$

Figure 3. d.c. representation for first example.

For this example, the functions $s_j(x)$ and $t_j(x)$ are given by

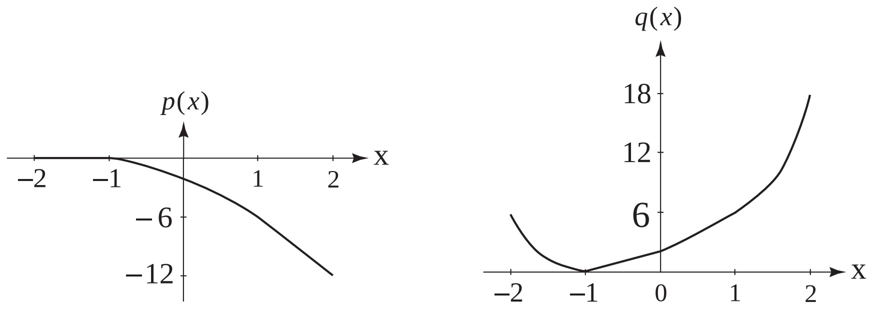
$$
\begin{array}{ll}
s_0(x) = \quad 0 & t_0(x) = \quad 0 \\
s_1(x) = \quad 6.263345 - 2.940585\cdot x & t_1(x) = \quad 6.263345 - 2.940585\cdot x \\
s_2(x) = -4.529474 + 2.455613\cdot x & t_2(x) = -10.792828 + 5.396197\cdot x \\
s_3(x) = \quad 2.131589 - 0.267318\cdot x & t_3(x) = \quad 12.924417 - 5.663516\cdot x \\
s_4(x) = -0.569305 + 0.271249\cdot x & t_4(x) = -13.493722 + 5.934765\cdot x
\end{array}
$$

Using the functions $t_j(x)$ given above and the function $f(x)$ given in (5), the concave function $p(x)$ and the convex function $q(x)$ are given by

$$
p(x) = \begin{cases} f(x) & \text{if } x \in D_1 \\ 6.263345 - 2.940585\cdot x & \text{if } x \in D_2 \\ f(x) + 10.792828 - 5.396197\cdot x & \text{if } x \in D_3 \\ 12.924417 - 5.663516\cdot x & \text{if } x \in D_4 \\ f(x) + 13.493722 - 5.934765\cdot x & \text{if } x \in D_5 \end{cases}
$$

$$
q(x) = \begin{cases} 0 & \text{if } x \in D_1 \\ f(x) - 6.263345 + 2.940585\cdot x & \text{if } x \in D_2 \\ -10.792828 + 5.396197\cdot x & \text{if } x \in D_3 \\ f(x) - 12.924417 + 5.663516\cdot x & \text{if } x \in D_4 \\ -13.493722 + 5.934765\cdot x & \text{if } x \in D_5 \end{cases}
$$

These two functions are graphed in Figure 3. Thus, the d.c. representation of the function shown in Figure 1 consists of the concave and convex functions shown in Figure 3.



(a) Concave function $p(x)$ (b) Convex function $q(x)$

Figure 4. d.c. representation for second example.

4.2. SECOND EXAMPLE

For the second example, the function $f(x)$ shown in Figure 2 is given by (7). For this function, the endpoints of the subintervals D_j are determined by the endpoints of the original interval $D = [-2\,,\,2]$ and by the points where $f(x)$ is 'nonsmooth'. In addition, the slopes Δ_j are taken as the minimum (resp. maximum) of $f'_-(x)$ and $f'_+(x)$ for j odd (resp. even). In the leftmost subinterval $f(x)$ is convex. Hence, $l = 2$. Together, this yields

$$\begin{array}{ll}
D_2 = [a_2\,,\,b_2] = [-2\,,\,-1] & \Delta_2 = \max\{-2\,,\,2\} = 2 \\
D_3 = [a_3\,,\,b_3] = [-1\,,\,0] & \Delta_3 = \min\{-1\,,\,1\} = -1 \\
D_4 = [a_4\,,\,b_4] = [0\,,\,0] & \Delta_4 = \max\{-1\,,\,1\} = 1 \\
D_5 = [a_5\,,\,b_5] = [0\,,\,1] & \Delta_5 = \min\{-2\,,\,2\} = -2 \\
D_6 = [a_6\,,\,b_6] = [1\,,\,2] &
\end{array}$$

For this example, the functions $s_j(x)$ and $t_j(x)$ are given by

$$\begin{array}{ll}
s_1(x) = 0 & t_1(x) = 0 \\
s_2(x) = 2 + 2{\cdot}x & t_2(x) = 2 + 2{\cdot}x \\
s_3(x) = 0 - 1{\cdot}x & t_3(x) = -2 - 3{\cdot}x \\
s_4(x) = 0 + 1{\cdot}x & t_4(x) = 2 + 4{\cdot}x \\
s_5(x) = 2 - 2{\cdot}x & t_5(x) = 0 - 6{\cdot}x
\end{array}$$

Using the functions $t_j(x)$ given above and the function $f(x)$ given in (7), the concave function $p(x)$ and the convex function $q(x)$ are given by

$$p(x) = \begin{cases} 0 & \text{if } x \in D_2 \\ f(x) - 2 - 2{\cdot}x & \text{if } x \in D_3 \\ -2 - 3{\cdot}x & \text{if } x \in D_4 \\ f(x) - 2 - 4{\cdot}x & \text{if } x \in D_5 \\ -6{\cdot}x & \text{if } x \in D_6 \end{cases}$$

$$q(x) = \begin{cases} f(x) & \text{if } x \in D_2 \\ 2 + 2{\cdot}x & \text{if } x \in D_3 \\ f(x) + 2 + 3{\cdot}x & \text{if } x \in D_4 \\ 2 + 4{\cdot}x & \text{if } x \in D_5 \\ f(x) + 6{\cdot}x & \text{if } x \in D_6 \end{cases}$$

These two functions are graphed in Figure 4. Thus, the d.c. representation of the function shown in Figure 2 consists of the concave and convex function shown in Figure 4.

An overview of the conversion procedure and possible extensions to it are discussed next.

5. Summary and extensions

This paper has described a method for converting a class of univariate, continuous functions (referred to as class 'piecewise-C^2' or 'P-C^2' functions) into d.c. functions (i.e., functions that can be represented as the sum of a concave function and a convex function). The class of functions is very broad covering many types of nonlinear, nonconvex, and/or 'nonsmooth' functions. The conversion procedure gives the explicit functional and numerical form of the concave and convex function that comprise the d.c. function representation of an arbitrary class P-C^2 function.

The principal computational burden in the conversion procedure lies in the determination of the subintervals of the domain of the function such that the function alternates between a concave function and a convex function on adjacent subintervals. Depending on the form of the function, the subintervals can be derived explicitly or determined numerically (see Appendix A). This conversion procedure can serve as a 'preprocessing' step in an optimization problem whose objective function separates into univariate P-C^2 functions.

The definition of a class P-C^2 function $f(x)$ can also be modified slightly to admit the possibility of $f(x)$ being discontinuous at one or both of the endpoints of the interval $D = [a, b]$ (see also [14]). For instance, consider the following modification of the function given in (5):

$$f(x) = \begin{cases} K & \text{if } x = 0 \\ 2 - e^{-10(x-2)^2} - e^{-0.1(x-5)^2} & \text{if } 0 < x \leqslant 10 \end{cases} \tag{14}$$

Let $f_+(a)$ denote the right-hand limit of the value of $f(x)$ as $x \to a$. If K does not equal $f_+(a)$, then $f(x)$ is discontinuous at $x = a$ and hence does not conform to the definition of a class P-C^2 function. However, the conversion procedure presented in Section 3 can still be applied to the function in (14) to yield a concave function $p(x)$ and a convex function $q(x)$ such that $f(x) = p(x)+q(x)$. There are two cases to consider: $K \leqslant f_+(a)$ and $K > f_+(a)$. In the first case, $f(x)$ remains a concave function on the leftmost subinterval $D_1 = [0\,, 1.777095]$. In this case, there are no changes to the analysis of the example conducted in Section 4.1 except that $f(x)$ is defined according to (14) rather than (5).

In the second case, we require the right-hand limit $f_+(a)$, and the right-hand derivative $f'_+(a)$ to be finite-valued (which is satisfied for (14)). We replace the subinterval $D_1 = [0\,, 1.777095]$ with two subintervals: $D_0 = [0\,, 0]$ and $D_1 = (0\,, 1.777095]$. Note that the index of the leftmost interval is now $l = 0$ and that subinterval D_1 is no longer closed. The other subintervals D_2 through D_5 remain the same as those given in Section 4.1. For the affine functions $s_j(x)$, we set $s_j(x) = 0$ for $j < l$ and we set

$$s_0(x) = f_+(a) + f'_+(a)\cdot(x - a) = 1.917915 - 0.082085\cdot x$$

The other affine functions $s_1(x)$ through $s_4(x)$ are the same as in Section 4.1. The change in the specification of $s_0(x)$ means that the functions $t_j(x)$ must be recalculated. However, no other changes are required in the conversion procedure.

In short, the conversion procedure described in this paper is applicable to a broad class of continuous functions as well as to a broad class of fixed-charge functions.

In closing, it is worthwhile pointing out two limitations to this procedure. First, the definition of a class P-C^2 function requires the left-hand and the right-hand derivatives ($f'_+(x)$ and $f'_+(x)$) to be finite. To see the effect of this limitation, consider the simple continuous function

$$f(x) = \sqrt[3]{x} \quad \text{for } x \in D = [-3\,, 3] \tag{15}$$

Clearly, $f(x)$ is convex on the leftmost subinterval indexed by $l = 2$ where $D_2 = [-3\,, 0]$ and concave on the rightmost subinterval indexed by $r = 3$ where $D_3 = [0\,, 3]$. But at $x = 0$, we have $f'_-(x) = +\infty$ and $f'_+(x) = +\infty$. This means that $\Delta_2 = +\infty$ and the affine function '$s_2(x) = \infty\cdot x$' is not defined. Thus, the cube-root function given by (15), which is not of class P-C^2, cannot be converted into a d.c. function using the procedures described in this paper.

The other limitation rests on the fact that class P-C^2 is defined for univariate functions. This definition can be extended to separable multivariate functions

$f : \mathbb{R}^n \to \mathbb{R}$ (where $f(\underline{\boldsymbol{x}}) = \sum_{i=1}^{n} f_i(x_i)$ and each $f_i(x_i)$ is a class P-C^2 function) or to indefinite quadratic multivariate functions (which can be transformed into separable functions [18]). However, there does not appear to be a straightforward way of extending the conversion procedure described in this paper to general *non-separable* multivariate functions. Hence, procedures for converting functions into d.c. functions remain an open and active area of research.

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Appendix A

The method for partitioning the interval $D = [a, b]$ into subintervals such that $f(x)$ alternates between a concave function and a convex function on adjacent subintervals is specific to the functional form of $f(x)$.

For the exponential function given in (5), the endpoints of the subintervals can be found by numerically determining the values of x such that $f''(x)$, the second derivative of $f(x)$, equals zero and $f'''(x)$, the third derivative of $f(x)$, is strictly different from zero. For the 'nonsmooth' function given in (7), the endpoints can be determined by inspection. In this appendix, we illustrate two additional methods. First, for a fourth-order polynomial, we show how the endpoints can be derived explicitly. Second, for a piecewise linear function, we present an algorithmic method for identifying the endpoints.

A.1. FOURTH-ORDER POLYNOMIAL FUNCTION

Let $f(x)$ be given by

$$f(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon \quad \text{for } x \in D = [a, b] \tag{16}$$

where α, β, γ, δ, and ϵ are coefficients. The points where $f(x)$ switches between a concave function and a convex function depend on the values of these coefficients. To specify these points, we define three constants – denoted by c, d, and e – as

$$c = \frac{-\gamma}{3\beta} \qquad d = \frac{-\beta - \sqrt{\beta^2 - \frac{8}{3}\alpha\gamma}}{4\alpha} \qquad e = \frac{-\beta + \sqrt{\beta^2 - \frac{8}{3}\alpha\gamma}}{4\alpha} \tag{17}$$

There are nine mutually exclusive, collective exhaustive possible cases to consider depending on the values of α, β, and γ. These nine cases are summarized in Table 1.

Table 1. Concave and convex subtintervals for fourth-degree polynomial

Case	Coefficient Conditions	Concave Subinterval	Convex Subinterval
1	$\alpha = 0$ & $\beta = 0$ & $\gamma = 0$	$-\infty \leqslant x \leqslant +\infty$	$-\infty \leqslant x \leqslant +\infty$
2	$\alpha = 0$ & $\beta = 0$ & $\gamma < 0$	$-\infty \leqslant x \leqslant +\infty$	Null set
3	$\alpha = 0$ & $\beta = 0$ & $\gamma > 0$	Null set	$-\infty \leqslant x \leqslant +\infty$
4	$\alpha = 0$ & $\beta < 0$	$x \geqslant c$	$x \leqslant c$
5	$\alpha = 0$ & $\beta > 0$	$x \leqslant c$	$x \geqslant c$
6	$\alpha < 0$ & $\beta^2 \leqslant \frac{8}{3}\alpha\gamma$	$-\infty \leqslant x \leqslant +\infty$	Null set
7	$\alpha < 0$ & $\beta^2 > \frac{8}{3}\alpha\gamma$	$x \leqslant d$ & $x \geqslant e$	$d \leqslant x \leqslant e$
8	$\alpha > 0$ & $\beta^2 \leqslant \frac{8}{3}\alpha\gamma$	Null set	$-\infty \leqslant x \leqslant +\infty$
9	$\alpha > 0$ & $\beta^2 > \frac{8}{3}\alpha\gamma$	$d \leqslant x \leqslant e$	$x \leqslant d$ & $x \geqslant e$

The number of subintervals D_j and determination of the endpoints these subintervals depends upon the location on the real number line of the constants c, d, and e with respect to a and b (the endpoints of the interval D). For instance, in case number 4 in Table 1, if $c \leqslant a$, then D is partitioned into a single (sub)interval $D_1 = [a_1, b_1] = [a, b]$ in which $f(x)$ is concave; if $c \geqslant b$, then D is partitioned into a single (sub)interval $D_2 = [a_2, b_2] = [a, b]$ in which $f(x)$ is convex; and if $a < c < b$, then D is partitioned into a subinterval $D_1 = [a_1, b_1] = [a, c]$ in which $f(x)$ is concave and a subinterval $D_2 = [a_2, b_2] = [c, b]$ in which $f(x)$ is convex. In a similar manner, the endpoints for each subinterval can be specified explicitly for all the cases given in Table 1.

A.2. PIECEWISE-LINEAR FUNCTION

Let $f(x)$ be a continuous function composed of m piecewise-linear segments in the interval $D = [a, b]$. For $k = 1, \dots, m$ let α_k be the slope of the k-th segment, and let β_k and β_{k+1} denote, respectively, the left and right endpoints of the k-th segment. By construction, $\beta_1 = a$ and $\beta_{m+1} = b$. Note that the number of subintervals will be less than or equal than m (the number of piecewise-linear segments). We assume (without loss of generality) that $f(x)$ is concave in the leftmost subinterval. Here, l (the index of the leftmost subinterval) is one; and r (the index of the rightmost subinterval) is equal to the number of subintervals.

The algorithm coded in Figure 5 can be used to determine the number of subintervals, r, and the endpoints a_j and b_j in the jth subinterval $D_j = [a_j, b_j]$ for $j = 1, \dots, r$ for any piecewise-linear function $f(x)$.

```
begin
    a_1 := β_1
    j := 1
    α_0 := +∞
    for k = 1 to m
        if (j odd and α_k > α_{k−1}) or (j even and α_k < α_{k−1})
            b_j := β_k
            a_{j+1} := β_k
            j := j + 1
        endif
    endfor
    b_j := β_{m+1}
    r := j
end
```

Figure 5. Pseudo code to determine endpoints for piecewise-linear function.

Appendix B

This appendix contains the proofs of the lemmas given in Section 3.

Proof of Lemma 1. To prove this lemma, we shall assume the contrary and show a contradiction. That is, we shall assume that there are an infinite number of subintervals satisfying Definition 2 in the original interval $[a, b]$. Since the definition of a class P-C^2 function requires that there are only a finite number of points in which $f''(x)$ is discontinuous, this implies that there is at least one interval, call it $[\overline{a}, \overline{b}]$, contained in $[a, b]$ such that $f''(x)$ is continuous in $[\overline{a}, \overline{b}]$ and there are an infinite number of subintervals in $[\overline{a}, \overline{b}]$. This means that $f''(x)$ must alternate between being strictly negative (when j is odd) and strictly positive (when j is even) an infinite number of times in the interval $[\overline{a}, \overline{b}]$. Let $\overline{x}_j$ be a point in the jth subinterval in $[\overline{a}, \overline{b}]$ such that $f''(\overline{x}_j)$ is strictly negative (resp., strictly positive) when j is odd (resp., even). Let $\overline{D}_j = [\overline{x}_j - \delta, \overline{x}_j + \delta]$ where δ is a constant. Because $f''(x)$ is continuous in $[\overline{a}, \overline{b}]$, there must exist a strictly positive value of δ such that $\overline{D}_j$ is contained in $[\overline{a}, \overline{b}]$ and $f''(x)$ is strictly different than zero for all $x \in \overline{D}_j$. By assumption, there are an infinite number of subintervals in $[\overline{a}, \overline{b}]$. This implies that for a strictly positive value of δ, there is an infinite number of nonoverlapping intervals $\overline{D}_j$ each of length 2δ contained in the interval $[\overline{a}, \overline{b}]$. This, in turn, implies that the length of the interval $[\overline{a}, \overline{b}]$ must be infinite. But this is a contradiction since $[\overline{a}, \overline{b}]$ is contained in the original interval $[a, b]$ which is of finite length. This contradiction means that the assumption of an infinite number of subintervals is incorrect. Therefore, there must be only a finite number of subintervals. □

Proof of Lemma 2. We show that $p(x)$ given in (12) is a concave function. An analogous procedure can be used to show that $q(x)$ given in (13) is convex. To show that $p(x)$ is concave, observe that $p(x)$ can be rewritten as

$$p(x) = \sum_{\substack{j=l \\ j \text{ odd}}}^{r} g_j(x)$$

where

$$g_j(x) = \begin{cases} 0 & \text{if } x < a_j \\ f(x) - s_{j-1}(x) & \text{if } a_j \leqslant x \leqslant b_j \\ s_j(x) - s_{j-1}(x) & \text{if } x > b_j \end{cases}$$

We now show that each $g_j(x)$ (for j odd) is a concave function on $[a\,,b]$. Note that for j odd, $g_j(x)$ is concave on each of the three regions $x < a_j$, $a_j \leqslant x \leqslant b_j$, and $x > b_j$. Let the left-hand (resp., right-hand) derivative of $g_j(x)$ be denoted by $g'_{j-}(x)$ (resp., $g'_{j+}(x)$). To show that $g_j(x)$ is concave on $[a\,,b]$, we need only verify that if $a_j > a$ then $g_j(x)$ is continuous at a_j and $g'_{j-}(a_j) \geqslant g'_{j+}(a_j)$; and if $b_j < b$ then $g_j(x)$ is continuous at b_j and $g'_{j-}(b_j) \geqslant g'_{j+}(b_j)$. To show that $g_j(x)$ is continuous at a_j note that

$$\begin{aligned} f(a_j) - s_{j-1}(a_j) &= f(b_{j-1}) - s_{j-1}(b_{j-1}) \\ &= f(b_{j-1}) - \big(f(b_{j-1}) - \Delta_{j-1}\cdot(b_{j-1} - b_{j-1})\big) = 0 \end{aligned}$$

and to show that $g_j(x)$ is continuous at b_j note that

$$f(b_j) - s_{j-1}(b_j) = f(b_j) - \Delta_j\cdot(b_j - b_j) - s_{j-1}(b_j) = s_j(b_j) - s_{j-1}(b_j)$$

Next, to show that the left-hand derivative is greater than or equal to the right hand derivative at a_j note that

$$g'_{j-}(a_j) = 0$$

$$g'_{j+}(a_j) = f'_+(a_j) - \Delta_{j-1} = f'_+(b_{j-1}) - \max\{f'_-(b_{j-1})\,,\, f'_+(b_{j-1})\} \leqslant 0$$

Hence, $g'_{j-}(a_j) \geqslant g'_{j+}(a_j)$. Similarly, to show that the left-hand derivative is greater than or equal to the right hand derivative at b_j note that

$$g'_{j-}(b_j) = f'_-(b_j) - \Delta_{j-1}$$

$$g'_{j+}(b_j) = \Delta_j - \Delta_{j-1}$$

Since $\Delta_j = \min\{f'_-(b_j)\,,\, f'_+(b_j)\} \leqslant f'_-(b_j)$, this means that $g'_{j-}(b_j) \geqslant g'_{j+}(b_j)$. Hence, $g_j(x)$ (for j odd) is concave on $[a\,,b]$. Now, since $p(x)$ is equal to a finite number of functions that are concave on $[a\,,b]$, this implies that $p(x)$ is also concave on $[a\,,b]$. *Q.E.D.*

Proof of Lemma 3. To show that $f(x) = p(x) + q(x)$ for all $x \in D$, we note simply that if $x \in D_j$ and j is odd, then

$$p(x) + q(x) = \big(f(x) - t_{j-1}(x)\big) + t_{j-1}(x) = f(x)$$

and that if $x \in D_j$ and j even, then

$$p(x) + q(x) = t_{j-1}(x) + \big(f(x) - t_{j-1}(x)\big) = f(x)$$

Since $\bigcup_{j=l}^{r} D_j = D$, we have $f(x) = p(x) + q(x)$ for all $x \in D$. □

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